

Chapter 15 Theories of Yielding

15.2 Maximum Principle Stress Theory

Theories of yielding are generally expressed in terms of principle stress, since those completely determine a general states of stress. The elements of material shown in Fig. 15.1(a) is subjected to three principle stresses, and the convention to be used is that $S_1 > S_2 > S_3$.

The *maximum principle* stress theory, often attributed to **Rankine**, states that yielding will occur in a material under complex stress when S_1 , in a simple tension test on the same material. Yielding could also occur if the minimum stress S_3 , were compressive and reached the value of yield stress in a simple compression test. Those statements may be written as

$$S_1 = S_{yt} \quad \text{or} \quad S_3 = S_{yc} \quad (15.1)$$

for yielding stress to occur.

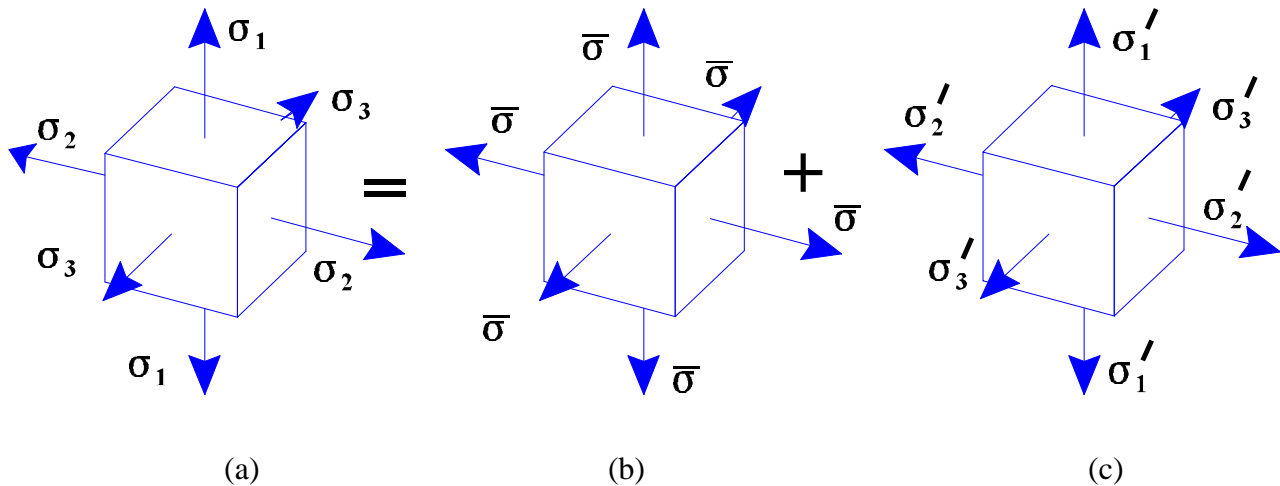


Fig. 15.1

15.3 Maximum Principle Strain Theory.

St.Venant postulated that yielding commences when the maximum *principle strain* (tensile), e_1 , was equivalent to the strain corresponding to the yield stress in simple tension. For yielding in compression the minimum principle strain, e_3 , would equal the yield strain in simple compression. If the strains are expressed in terms of stress, then

$$e_1 = \frac{s_1}{E} - \frac{n}{E}(s_2 + s_3)$$

and yielding occurs when compression equals $\frac{s_{Yt}}{E}$:

$$s_1 - n(s_2 + s_3) = s_{Yt} \quad (15.2)$$

or for compression,

$$s_3 - n(s_1 + s_2) = s_{Yc} \quad (15.3)$$

15.4 Maximum Shear Stress Theory.

The assumption in this theory is that yielding is dependent on the maximum shear stress in the material reaching a critical value. This latter value is taken as the maximum shear stress in a simple

tensile test, which is half the yield stress, or $\frac{1}{2}s_{Yt}$. The maximum shear stress in the complex stress system will depend on the relative values and sign of the three principle stresses, always being half the difference between the maximum and minimum.

For general three-dimensional stress system, or in the two-dimensional case with one of the stresses compressive and the other tensile, the maximum shear stress is.

$$\hat{t} = \frac{s_1 - s_3}{2} \quad \text{And for yielding} \quad \frac{s_1 - s_3}{2} = \frac{s_{Yt}}{2} \quad \text{or} \quad s_1 - s_3 = s_{Yt} \quad (15.4)$$

In a two-dimensional stress system when $s_3 = 0$, ie for s_1 and s_2 tensile, the maximum difference between the principle stress is

$$\hat{t} = \frac{s_1 - 0}{2} = \frac{s_1}{2} \quad \text{And yielding occurs if} \quad \frac{s_1}{2} = \frac{s_{Yt}}{2} \quad \text{or} \quad s_1 = s_{Yt} \quad (15.5)$$

This theory is usually coupled with the names of **Guest** and **Tresca**.

15.5 Total Strain energy Theory

The theories put forward so far have postulated a criterion for yielding in terms of a limit value of

stress or strain. The present theory, as proposed by **Beltrami**, and also attributed to **Haigh**, is based on a critical value of the **total strain energy** stored in the material, and this is a product of stress and strain.

It has been shown earlier that the work done in deformation or the stored elastic strain energy may

be written as $\frac{1}{2}Wdx$ or

$$\frac{\frac{1}{2}Wdx}{Ax} = \frac{1}{2}\mathbf{se} \quad \text{per unit volume}$$

In a three-dimensional stress system, the total strain energy is

$$U_T = \frac{1}{2}\mathbf{s}_1\mathbf{e}_1 + \frac{1}{2}\mathbf{s}_2\mathbf{e}_2 + \frac{1}{2}\mathbf{s}_3\mathbf{e}_3$$

Now using a stress-strain relationship, the principle strains may be written as

$$\mathbf{e}_1 = \frac{\mathbf{s}_1}{E} - \frac{\nu}{E}(\mathbf{s}_2 + \mathbf{s}_3)$$

$$\mathbf{e}_2 = \frac{\mathbf{s}_2}{E} - \frac{\nu}{E}(\mathbf{s}_3 + \mathbf{s}_1)$$

$$\mathbf{e}_3 = \frac{\mathbf{s}_3}{E} - \frac{\nu}{E}(\mathbf{s}_2 + \mathbf{s}_1)$$

substituting for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and rearranging,

$$U_T = \frac{1}{2E}(\mathbf{s}_1^2 + \mathbf{s}_2^2 + \mathbf{s}_3^2) - \frac{n}{2E}(\mathbf{s}_1\mathbf{s}_2 + \mathbf{s}_2\mathbf{s}_3 + \mathbf{s}_3\mathbf{s}_1)$$

Yielding is said to occur when the above is equal to the total strain energy at yield in simple tension,

ie. By putting $\mathbf{s}_2 = \mathbf{s}_3 = 0$ and $\mathbf{s}_1 = \mathbf{s}_{Yt}$,

$$\text{Therefore } \mathbf{s}_1^2 + \mathbf{s}_2^2 + \mathbf{s}_3^2 - 2n(\mathbf{s}_1\mathbf{s}_2 + \mathbf{s}_2\mathbf{s}_3 + \mathbf{s}_3\mathbf{s}_1) = \mathbf{s}_{Yt}^2 \quad (15.6)$$

$$U_T = \frac{\mathbf{S}_{Yt}^2}{2E}$$

In the two-dimensional system, $\mathbf{S}_3 = 0$ and

$$\mathbf{S}_1^2 + \mathbf{S}_2^2 - 2n\mathbf{S}_1\mathbf{S}_2 = \mathbf{S}_{Yt}^2 \quad (15.7)$$

15.6 Shear or Distortion Strain Energy Theory

Huber, in 1904, proposed that the total strain energy of an element of material could be divided into two parts, that due to change in volume and that due to change in shape. These will be termed *volumetric stain energy* U_v , and *distortion or shear strain energy*, U_s . It is rather more simple to determine the former quantity than the latter, and since the total strain energy has already been determined, the shear or distortion component can be determined as

$$U_s = U_T - U_v \quad (15.8)$$

In order to show that the deformation of a material can be separated into change in volume and change in shape, consider the element in fig.15.1 subjected to the principle stress $\mathbf{S}_1, \mathbf{S}_2$ and \mathbf{S}_3 . Those may be written in terms of the “average” stress in the element as follows:

$$\left. \begin{aligned} \mathbf{S}_1 &= \bar{\mathbf{S}} + \mathbf{S}'_1 \\ \mathbf{S}_2 &= \bar{\mathbf{S}} + \mathbf{S}'_2 \\ \mathbf{S}_3 &= \bar{\mathbf{S}} + \mathbf{S}'_3 \end{aligned} \right\} \quad (15.9)$$

Where $\bar{\mathbf{S}}$ is the average or mean stress defined as

$$\bar{\mathbf{S}} = \frac{\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3}{3} \quad (15.10)$$

Now, when an element as in Fig.15.1(b) is subjected to $\bar{\mathbf{S}}$ in all directions, this hydrostatic stress will produce a change in volume, but no distortion. Consider the effect of the \mathbf{S}' component of stress. Adding together equations.(15.9) gives.

$$\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 = 3\bar{\mathbf{S}} + \mathbf{S}'_1 + \mathbf{S}'_2 + \mathbf{S}'_3$$

but $\bar{s} = \frac{1}{3}(s_1 + s_2 + s_3)$; hence

$$s'_1 + s'_2 + s'_3 = 0 \quad (15.11)$$

But from the stress-strain relationship

$$\left. \begin{aligned} e'_1 &= \frac{s'_1}{E} - \frac{n}{E}(s'_2 + s'_3) \\ e'_2 &= \frac{s'_2}{E} - \frac{n}{E}(s'_3 + s'_1) \\ e'_3 &= \frac{s'_3}{E} - \frac{n}{E}(s'_1 + s'_2) \end{aligned} \right\} \quad (15.12)$$

Hence

$$e'_1 + e'_2 + e'_3 = e'_v = \frac{(1-2n)}{E}(s'_1 + s'_2 + s'_3) \quad (15.13)$$

and since the sum of the three stresses is zero, eqn.(15.11)

$$e'_1 + e'_2 + e'_3 = e'_v = 0 \quad (15.14)$$

Thus the stress components can cause no change in volume but only change in shape.

The volumetric strain can now be determined from hydrostatic component of stress, \bar{s} .

$$\begin{aligned} U_v &= \frac{1}{2} \bar{s} e \\ &= \frac{1}{2} \bar{s} \frac{3\bar{s}}{E} (1-2n) \\ &= \frac{1}{2} \left[\frac{s_1 + s_2 + s_3}{3} \right] \left[\frac{3(1-2n)}{E} \right] \left[\frac{s_1 + s_2 + s_3}{3} \right] \end{aligned} \quad (15.15)$$

$$= \frac{1-2n}{6E} (s_1 + s_2 + s_3)^2$$

But $U_s = U_T - U_v$; therefore;

$$U_s = \frac{1}{2E} [s_1^2 + s_2^2 + s_3^2 - 2n(s_1s_2 + s_2s_3 + s_3s_1)] - \frac{1-2n}{6E} (s_1 + s_2 + s_3)^2$$

which reduces to

$$U_s = \frac{1+2n}{6E} [(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2] \text{ per unit volume} \quad (15.16)$$

or alternatively, using the relationship between E, G and n ,

$$U_s = \frac{1}{12G} [(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2] \quad (15.17)$$

Now, the shear or distortion strain energy theory proposes that yielding commences when the quantity U_s reaches the equivalent value at yielding in simple tension. In the latter case s_2 and $s_3 = 0$ and $s_1 = s_{yt}$ therefore

15.6 Shear or Distortion Strain Energy Theory

$$U_s = \frac{s_{yt}^2}{6G} \text{ energy per unit volume} \quad (15.18)$$

$$\frac{1}{12G} [(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_{31} - s_1)^2] = \frac{s_{yt}^2}{6G}$$

or (15.19)

$$(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_{31} - s_1)^2 = 2s_{yt}^2$$

In the two-dimensional system $s_3 = 0$ and

$$s_1^2 + s_{21}^2 - s_1s_2 = s_{\Psi t}^2 \quad (15.20)$$

This theory was also independently established by **Maxwell, von Mises and Hencky**, and is generally referred to as the **Mises criterion**.

The above analysis has been directly aimed at establishing a yield criterion on an energy basis. However, one might equally well propose that yielding occurred as a function of difference between principle stresses. On this hypothesis it is evident that eqn.(15.19) is also obtained by considering the root mean square of the principle stress difference in the complex stress system in relation to simple tension. Thus;

$$\sqrt{\left[\frac{1}{3}\{(a_1 - s_2)^2 + (a_2 - s_3)^2 + (a_3 - s_1)^2\}\right]} = \sqrt{\left[\frac{1}{3}(2s_{\psi_t}^2)\right]} \quad (15.21)$$

The right-hand side of the equation is obtained for simple tension by putting $s_1 = s_{\psi_t}$ and

$$s_2 = s_3 = 0 . \text{ Squaring both sides of eqn.(15.21).}$$

$$(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2 = 2s_{\psi_t}^2$$

which is the same as eqn.(15.19)

Summary

Many experiments have been concluded under complex stress conditions to study the behaviour of metals and test the validity of the foregoing theories. It has been shown that hydrostatic pressure, and by inference hydrostatic tension, does not cause yielding. Now any complex stress system can be regarded as a combination of hydrostatic stress and a function of the differences of principle stress, and therefore a yield criterion such as that of **Tresca or von Mises** which is based on principle stress difference would seem to be the most logical. It is now well established that for ductile metals, exhibiting yielding and subsequent plastic deformation, the shear strain energy theory correlates best with material behaviour.

The maximum shear stress theory, although not quite so consistent as the former, gives fairly reasonable prediction and is some times used in design by virtue of its simpler mathematical form. **The other theories are no longer used for ductile metals, some being positively unsafe.**

15.7 Yield Loci

The expression of the five theories that have been discussed can be plotted graphically for the simplified conditions of $s_3 = 0$ and equal yield stress in simple tension and compression. In Fig.15.2 co-ordinate axes of the principle stresses s_1 and s_2 are drawn and the curves shown represents the five theories plotted on this basis. The maximum principal stress theory is represented by the square

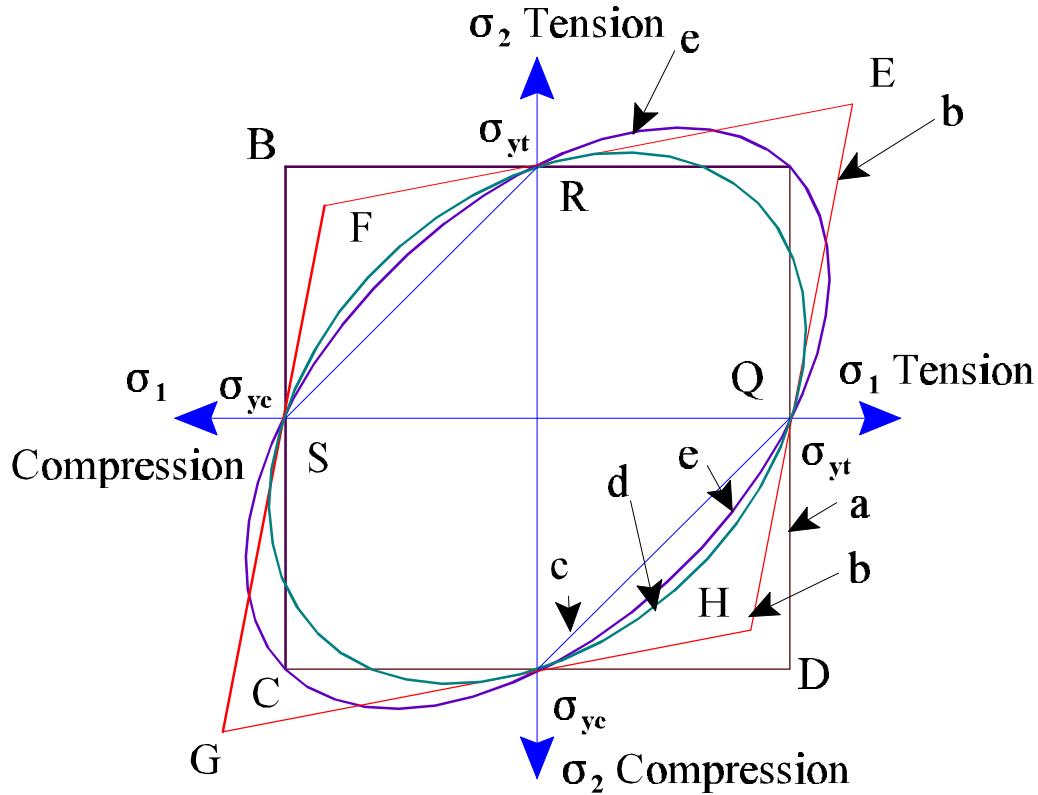
ABCD, and accordance to this theory condition inside the boundaries of the square will be elastic, while outside the area, yield will have occurred. The *principle strain theory gives the rhombus EFGH*, the co-ordinates of these points being.

$$\left(\frac{\mathbf{S}_{Yt}}{1-n}, \frac{\mathbf{S}_{Yt}}{1-n}\right) \quad \left(\frac{\mathbf{S}_{Yc}}{1+n}, \frac{\mathbf{S}_{Yt}}{1+n}\right) \quad \text{Respectively}$$

$$\left(\frac{\mathbf{S}_{Yc}}{1-n}, \frac{\mathbf{S}_{Yc}}{1-n}\right) \quad \left(\frac{\mathbf{S}_{Yt}}{1+n}, \frac{\mathbf{S}_{Yc}}{1+n}\right)$$

For principle stress of opposite sign, the maximum shear stress theory has yield lines in the second and fourth quadrants denoted by **RS** and **QT**. However, in the first and third quadrants, where \mathbf{S}_1 and \mathbf{S}_2 are of like sign $\mathbf{S}_3 = 0$ has to be used to give the maximum shear stress, and the yield boundary coincides with the maximum principle stress theory at QAR and SCT. The two energy criteria plot as ellipses, and the ellipse due to the shear strain energy criterion circumstances the *maximum shear stress hexagon*.

Each of those diagrams is termed a yield locus, and as stated previously, inside the locus elastic conditions prevail. The locus itself represents the onset of yielding, and outside the locus the material is in the plastic range. It is seen that, except for the small areas cut off at the corners A and C, the maximum shear stress criterion is most conservative of the five loci.



- a = Maximum principle stress theory
- b = Maximum principle strain theory
- c = Maximum shear stress theory
- d = Total strain energy theory
- e = Shear strain energy theory

Fig 15.2

A diagram such as Fig.15.2 serves several useful purposes. Experimental points can be plotted on it and compared with each of the theoretical curves as in Fig.19.16. The latter can readily be assessed against each other, and ratios of S_1 to S_2 to cause yielding can be quickly determined. For example,

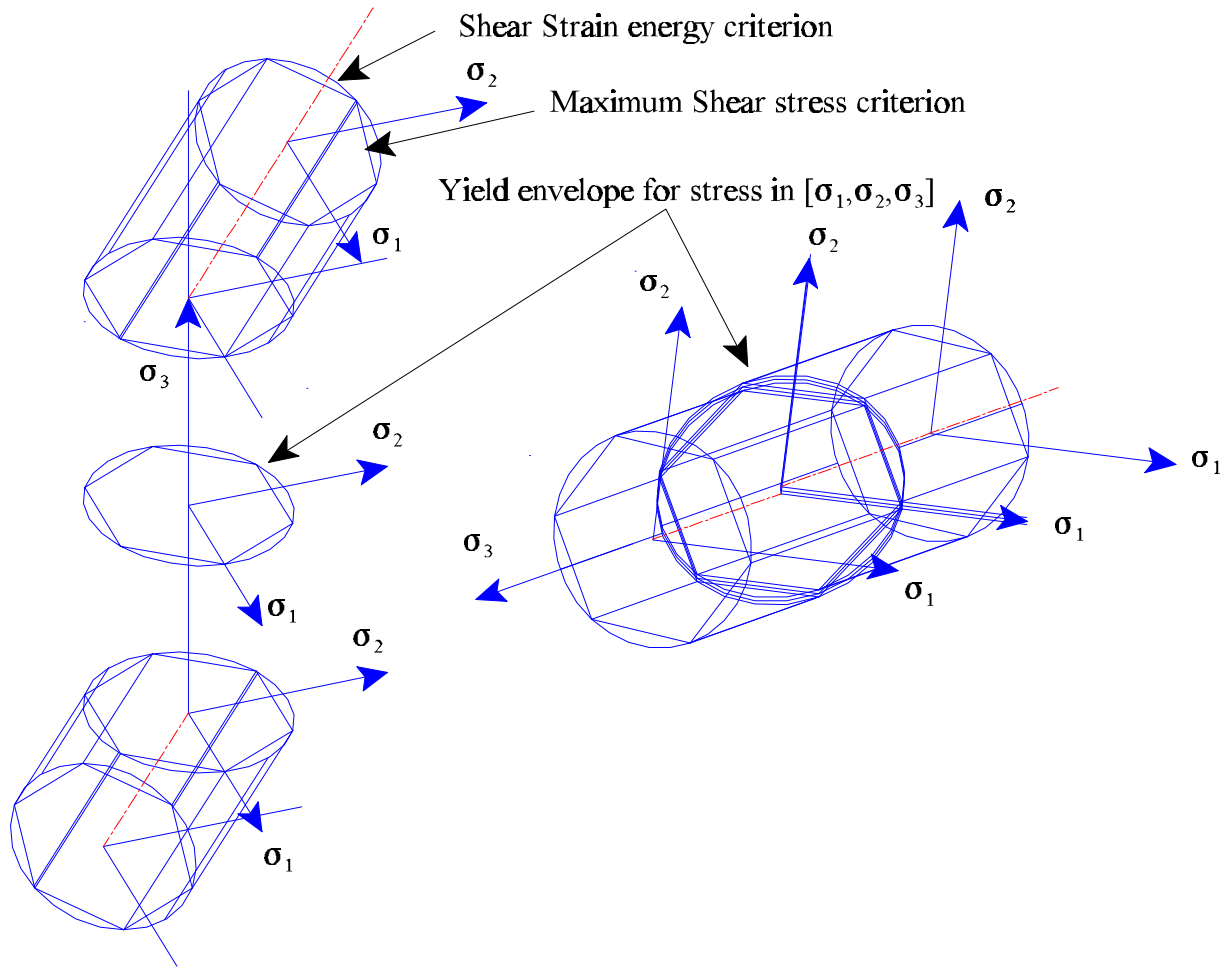
in case of equal biaxial tension (or compression), $\frac{S_1}{S_2} = 1$, and processing along this line from the

origem the loci are reached in order of theories (d), (a,c,e) and (b), whereas for pure torsion,

$\frac{S_1}{S_2} = -1$, and other order of yield boundaries become (c),(e),(d),(b),(a).

For the case of three principle stresses, all non-zero, the yield locus becomes a yield envelope centred on co-ordinate axis S_1, S_2, S_3 . The maximum shear stress and shear strain energy theories are represented by hexagonal and circumscribing circular cylinders respectively as illustrated in Fig.15.3.

The axis of the cylinders is equally inclined to each of the three co-ordinate axis. Inside the cylinders an elastic conditions exist, while outside the material is in the plastic range.



15.8 Brittle Materials

Brittleness in a material may be defined as absence of the ability to deform plastically. Materials such as a flake cast-iron, concrete and ceramics when subjected to tensile stress will generally fracture at the elastic limit or only a very small strain beyond this point. This means that the term yield criterion used for ductile materials may often also imply a fracture criterion for brittle material

Experiment has shown that the maximum principle stress theory is most satisfactory for predicting failure.

Some Brittle materials are considerable stronger in compression then tension, and **Mohr** forward a construction circle, to allow for this in the application of maximum shear stress theory.